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GROWTH OF SPHERICAL GAS BUBBLES BY SOLUTE DIFFUSION IN NON-NEWTONIAN (POWER LAW) LIQUIDS

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Abstract—In this work, diffusive bubble growth in newtonian liquids and non-Newtonian liquids of the power law type is studied. Three regions are considered, the first where growth is limited entirely by the rate of arrival dissolved gas at the bubble surface, the second where growth is limited entirely by hydrodynamic forces and the third, a general case, where both diffusion and hydrodynamics play a part. Parameters which have previously been used to define regions where viscosity, inertia, surface tension and diffusion are important are shown to hold good for power law liquids. It is found that, all else being equal, bubbles in power law liquids grow more slowly the smaller the characteristic exponent.

NOMENCLATURE

a,	dimensionless radial co-ordinate distance;
A,	dimensionless bubble radius;
a_b,	instantaneous area of a bubble;
B,	$= Ja^2/G^{1/2}$, growth regime characterisation parameter;
$\bar{B}$,	$= \overline{Ja}^2/G^{1/2}$, reference value of B;
C,	concentration of dissolved gas;
C_0,	concentration in equilibrium with P_0;
C_{sat},	concentration in equilibrium with P_∞;
C^*,	$= (C - C_{sat})/(C_0 - C_{sat})$, dimensionless concentration;
ΔC,	$= C_0 - C_{sat}$;
D,	diffusion coefficient;
g,	acceleration due to gravity;
G,	$= R_0^2 \Delta P/D^2 \rho_l$, inertial characterisation parameter;
H,	Henry's law coefficient;
Ja,	$= \Delta C/\rho_g$, Jakob number;
$\overline{Ja}$,	$= (\Delta C/\rho_g)_{P_\infty}$, reference value of the Jakob number;
m,	$= m'/\rho_l$, "kinematic viscosity coefficient" for a power law liquid;
m',	coefficient in power law;
n,	power law liquid exponent;
P,	pressure;
P_0,	initial pressure, saturated bubble;
P_g,	pressure inside bubble;
P_∞,	ambient pressure;
ΔP,	$= P_0 - P_\infty$;
r,	radial co-ordinate distance;
R,	bubble radius;
R_0,	initial bubble radius;
$\dot{R}$,	velocity of bubble surface;
$\ddot{R}$,	acceleration of bubble surface;
Re,	Reynolds number;
s,	spacing parameter;
S,	similarity transformation;
Sc,	$= \nu/D$ Schmidt number, Newtonian liquid;
Sc^*,	$= mR_0^{2(1-n)}/D^{(2-n)}$ Schmidt number, power law liquid;
Sh,	$= 2RK_l/D$, Sherwood number;
t,	time;
T,	$= (\frac{2}{3}G)^{1/2}\tau$, a dimensionless time;
v,	velocity;
V,	$= Sc/G^{1/2}$, viscous/inertia parameter, Newtonian;
V^*,	$= Sc^*/G^{[1-(n/2)]}$, as above, power law liquid;
x,	$= a - A$, dimensionless distance beyond the bubble surface;
y,	$= x/(s + x)$ spatial transformation variable.

Greek symbols

β,	dimensionless growth coefficient;
γ,	$= \Delta C/C_{sat}$, supersaturation driving force;
Δ,	rate of deformation tensor;
Δ_{ij},	component of Δ;
ε,	$= 1 - (\rho_g/\rho_l)$;
η,	scalar function of power law "viscosity";
μ,	viscosity coefficient;
ν,	$= \mu/\rho_l$, kinematic viscosity;
Π,	$= (P_g - P_\infty)/(P_0 - P_\infty)$, dimensionless pressure;
σ,	surface tension;
τ,	$= Dt/R_0^2$, dimensionless time;
τ,	shear stress tensor;
τ_{ij},	component of τ;
$\Phi\{\varepsilon, \beta\}$,	driving force parameter for bubble growth;
ϕ,	$= 2\sigma/\Delta P R_0$, surface tension parameter.

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Subscripts and superscripts

g,	gas phase;
i,	interface;
l,	liquid phase;
r,	in radial direction;
R,	at bubble surface;
θ,	in θ direction;
ϕ,	in ϕ direction;
∞,	at a distance or ambient;
$*$,	for a power law liquid;
$\cdot$,	differentiation with respect to time.

INTRODUCTION

IN THE last two decades there have been substantial and significant advances made in the study of bubble dynamics and phase growth. Much of the work has been directed towards a better understanding of the nucleate boiling process, prompted by the development of high flux heat exchange devices such as nuclear reactors and high pressure steam generating systems. In addition there are many problems in which mass transfer plays an important role, and it would be desirable to understand the behaviour of gaseous impurities dissolving in or evolving from liquids such as water or molten glass, the evolution of bubbles in vacuum degassing processes (such as that of degassing a steel melt), the growth of bubbles in the dissolved air flotation process, the determination of coefficients of diffusivity by observation of evolving or dissolving gas bubbles, the foaming of liquid plastics by gaseous evolution and the rate of solution of dispersed air bubbles in processes for BOD (biological oxygen demand) reduction in pollution control.

Most previous work in bubble dynamics has been applied to nucleate boiling. Scriven [1] gave a useful similarity solution, equally applicable to mass or thermal diffusion. He showed that, for only diffusional resistances, the relationship between bubble radius and the square root of time for a growing bubble, is a linear one. However, where other factors cause growth rate limitation, Scriven's simple solution is not applicable.

In a supersaturated gaseous solution, a bubble initiated by whatever means will grow as solute diffuses from the surroundings. At first the growth of the bubble depends on the inertia, viscosity and surface tension of the surrounding liquid. However, the growth will eventually become limited by the rate at which material can diffuse to the bubble surface.

Szekely and Martins [2] and Szekely and Fang [3] did much to rectify the earlier lack of attention paid to mass diffusional bubble growth. By using various dimensionless characterisation they were able to define regions where growth was limited by diffusion, by surface tension, by inertia or by the surface kinetics at the diffusional interface. This particular problem has also been considered in detail by Rosner and Epstein [4], who introduced generalized solution kinetics. Thus they assumed cases where the mass-transfer rate depended on differences in the concentrations of the solute gas raised to some general power n, rather than simply on the concentration difference alone.

Bubble growth in the non-Newtonian liquid has now been closely examined along the same lines [5]. Investigation of the growth of gas bubbles in non-Newtonian liquids of the power law type was undertaken because of the industrial importance of this class of material. Indeed plastics are often foamed by the nucleation and growth of bubbles of a diffusive substance. It is possible also that information on the normal stress behaviour in some non-Newtonian fluids could be obtained by observation on bubble growth. It is the object of this paper to present the results for the particular case of bubble growth in a fluid which obeys the Ostwald-de-Waele (power law) model of non-Newtonian behaviour.

FORMULATION

A spherical gas bubble is growing in a quiescent supersaturated liquid of infinite extent. The bubble is growing because of the transfer of dissolved gas from the liquid phase to the gas phase. The rate of growth is determined by the difference between the pressure within the bubble and the ambient pressure; liquid inertia, viscosity and surface tension; the diffusivity of the dissolved gas in the liquid and the degree of supersaturation of the dissolved gas. The liquid phase is assumed to be initially homogeneous and the gas phase within the bubble is assumed to be at a uniform pressure and homogeneous. The gas phase is in thermodynamic equilibrium with the liquid phase at the gas–liquid interface. The bubble maintains a spherical shape and the centre of the bubble remains fixed and is the centre of a spherically symmetric co-ordinate system.

Under the above assumptions and in the case where the rate of change of the density of the gas within the bubble is much less than the rate of change of the volume of the bubble the equations of continuity and momentum take the following forms:

$$v_r r^2 = \varepsilon \dot{R} R^2 \tag{1}$$

$$\rho_l \left[\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} \right] = \frac{\partial p_{rr}}{\partial r} + \frac{\tau_{\theta\theta} + \tau_{\phi\phi} - 2\tau_{rr}}{r}. \tag{2}$$

Further, the diffusion equation, combined with the continuity equation (1) and a diffusional mass balance at the interface become:

$$\frac{\partial c}{\partial t} = D\left[\frac{\partial^2 c}{\partial r^2} + \frac{2}{r}\frac{\partial c}{\partial r} \right] - \frac{\varepsilon \dot{R} R^2}{r^2} \frac{\partial c}{\partial r} \tag{3}$$

$$\rho_g \dot{R} = c(R, t)(1 - \varepsilon)\dot{R} + D \left. \frac{\partial c}{\partial r} \right|_R . \tag{4}$$

Initial and boundary conditions follow from the assumptions above and the specification that the bubble has an initial size and a zero initial growth rate. They are:

$$R = R_0 \quad \text{at} \quad t = 0 \tag{5}$$

$$\dot{R} = 0 \quad \text{at} \quad t = 0 \tag{6}$$

$$C = C_0 \quad \text{at} \quad t = 0,\ r > R \tag{7}$$

$$C = C_0 \quad \text{at} \quad r = \infty, \ \text{all } t. \tag{8}$$

$$P_g = HC|_R. \tag{9}$$

Three main classes of solution are now possible. Firstly, in the case where growth is limited solely by liquid properties of inertia, viscosity and surface tension, the momentum equation (2) is solved. Secondly, where liquid properties have no effect and the bubble growth rate is entirely determined by the rate of arrival of diffusant at the surface, the diffusion equation (3) is solved. Thirdly, where both liquid properties and diffusion provide resistances to bubble growth, equations (2) and (3) are solved simultaneously, coupled through the mass balance equation (4) and the equilibrium condition equation (9).

All the equations above apply equally to Newtonian and non-Newtonian liquids. The momentum equation (2) must firstly be put into a different form for solution.

The power law is a two parameter rheological model of the form:

$$\tau_{yx} = -m'\left|\frac{\mathrm{d}v_x}{\mathrm{d}y}\right|^{n-1}\frac{\mathrm{d}v_x}{\mathrm{d}y}. \tag{10}$$

For the case of $n = 1$, it reduces to Newton's law of viscosity with $m = \mu$. The deviation of n from unity indicates the degree of departure from Newtonian behaviour. For n less than unity, the behaviour is termed pseudoplastic; for n greater than unity, dilatant.

Bird, Stewart and Lightfoot [6] describe how the power law can be put into the following form:

$$\tau = -\{m'|\sqrt{\tfrac{1}{2}(\Delta:\Delta)}|^{n-1}\}\Delta \tag{11}$$

with components of the stress tensor τ being:

$$\tau_{rr} = -2\eta\frac{\partial v_r}{\partial r} \tag{12}$$

and

$$\tau_{\theta\theta} = \tau_{\phi\phi} = -2\eta\frac{v_r}{r} \tag{13}$$

where

$$\eta = m'\left[2\left\{\left(\frac{\partial v_r}{\partial r}\right)^2 + 2\left(\frac{v_r}{r}\right)^2\right\}\right]^{(n-1)/2}. \tag{14}$$

Combining equations (1) and (2) and (11)–(14), the momentum equation for a power law liquid becomes

$$\rho_l\left[\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r}\right] = \frac{\partial p_{rr}}{\partial r} - \frac{12m'k(2)^{(n-1)/2}}{r^4} \times\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + 2\left(\frac{v_r}{r}\right)\right]^{(n-1)/2}. \tag{15}$$

Equation (15) may be integrated from the bubble surface to infinity and after substitution of the normal stresses at the surface and at infinity, the following equation results:

$$\frac{P_g - P_\infty}{\varepsilon\rho_l} = R\ddot{R} + \left(1 + \frac{\varepsilon}{2}\right)\dot{R}^2 + \frac{12^{(n+1)/2}\varepsilon^{n-1}m}{3n}\left[\frac{\dot{R}}{R}\right]^n + \frac{2\sigma}{\varepsilon\rho_l R}. \tag{16}$$

For the Newtonian liquid where $n = 1$, the equivalent expression is:

$$\frac{P_g - P_\infty}{\varepsilon\rho_l} = R\ddot{R} + \left(1 + \frac{\varepsilon}{2}\right)\dot{R}^2 + 4\nu\frac{\dot{R}}{R} + \frac{2\sigma}{\varepsilon\rho_l R}. \tag{17}$$

A solution to these equations can be made more general if they are expressed in dimensionless form. Thus we define the following non-dimensional variables:

radius $A = R/R_0$ (18)

time $\tau = Dt/R_0^2$ (19)

pressure $\Pi = (P_g - P_\infty)/(P_0 - P_\infty)$, (20)

concentration $C^* = (C - C_{sat})/(C_0 - C_{sat})$ (21)

and the dimensionless group

$$G = R_0^2\Delta P/D^2\rho_l \tag{22}$$

which represents the relative importance of the liquid inertia and the driving pressure difference. We define also a surface tension parameter

$$\phi = 2\sigma/\Delta PR_0 \tag{23}$$

which indicates the degree of displacement of the initial radius from the equilibrium radius. If ϕ is less than unity the bubble will grow; if greater than unity it will dissolve. When $\phi = 1$, the bubble is in equilibrium with the surrounding liquid.

The Schmidt number for the Newtonian and power law fluids are, respectively,

$$Sc = \nu/D; \quad Sc^* = mR_0^{2(1-n)}/D^{2-n} \tag{24}$$

and the Jakob number is

$$Ja = (C_0 - C_{sat})/\rho_g. \tag{25}$$

The equations of bubble growth can now be presented in terms of these variables, for the case when $\varepsilon = 1$.

Momentum, Newtonian liquid:

$$A\frac{\mathrm{d}^2A}{\mathrm{d}\tau^2} + \frac{3}{2}\left(\frac{\mathrm{d}A}{\mathrm{d}\tau}\right)^2 + 4Sc\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}\tau} = G\left[\Pi - \frac{\phi}{A}\right]. \tag{26}$$

Momentum, power law liquid:

$$A\frac{\mathrm{d}^2A}{\mathrm{d}\tau^2} + \frac{3}{2}\left(\frac{\mathrm{d}A}{\mathrm{d}\tau}\right)^2 + \frac{12^{(n+1)/2}}{3}Sc^*\left(\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}\tau}\right)^n = G\left(\Pi - \frac{\phi}{A}\right). \tag{27}$$

Diffusion:

$$\frac{\partial c^*}{\partial\tau} = \frac{\partial^2 c^*}{\partial a^2} + \left\{\frac{2}{a} - \frac{A^2}{a^2}\frac{\mathrm{d}A}{\mathrm{d}\tau}\right\}\frac{\partial c^*}{\partial a}. \tag{28}$$

Initial and boundary conditions:

$$A = 1 \qquad \tau = 0 \tag{29}$$

$$\dot{A} = 0 \qquad \tau = 0 \tag{30}$$

$$C^* = 1 \qquad \tau = 0\colon\ a \geqslant 1 \tag{31}$$

$$C^* = 1 \qquad a = \infty\colon\ \text{all } \tau \tag{32}$$

$$\left.\frac{\partial c}{\partial a}\right|_A = \frac{1}{Ja}\frac{\mathrm{d}A}{\mathrm{d}\tau} \qquad a = A\colon\ \text{all } \tau \tag{33}$$

$$\Pi = C^*|_A \qquad a = A\colon\ \text{all } \tau. \tag{34}$$

NUMERICAL SOLUTIONS

There is no known analytical solution for equations (26)–(28). However they are readily tackled by numerical means of solution.

The three main classes of solution described above will be discussed in turn.

(a) *Growth limited by hydrodynamic forces*

When growth is limited by hydrodynamic forces only, diffusion to the bubble surface is relatively so rapid that at all times the concentration of the diffusing species at the surface is equal to the initial value. Thus the dimensionless pressure within the bubble, Π, of equations (26) and (27) is equal to unity through the whole growth period. The initial conditions equations (29) and (30) are appropriate.

These momentum equations now contain the three parameters G, ϕ and Sc or Sc^* as well as the two variables A and τ. The number may be reduced from five to four by making a change of variable of the kind $T \propto (G)^{1/2}\tau$. For consistency with the work of others [2, 6] a constant of 2/3 will be included so the following substitution will be made:

$$T = (\tfrac{2}{3}G)^{1/2}\tau. \tag{35}$$

This converts equations (26), (27), (29), (30) to:

Momentum, Newtonian liquid:

$$A\frac{\mathrm{d}^2A}{\mathrm{d}T^2}+\frac{3}{2}\left(\frac{\mathrm{d}A}{\mathrm{d}T}\right)^2+4\left(\frac{3}{2}\right)^{1/2}V\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}T} = \frac{3}{2}\left(1-\frac{\phi}{A}\right). \tag{36}$$

Momentum, power law liquid:

$$A\frac{\mathrm{d}^2A}{\mathrm{d}T^2}+\frac{3}{2}\left(\frac{\mathrm{d}A}{\mathrm{d}T}\right)^2+\frac{2^{3n/2}3^{1/2}}{n}V^*\left|\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}T}\right|^n = \frac{3}{2}\left(1-\frac{\phi}{A}\right). \tag{37}$$

Initial conditions:

$$A = 1 \qquad T = 0 \tag{38}$$

$$A = 0 \qquad T = 0 \tag{39}$$

where

$$V = Sc/G^{1/2} \tag{40}$$

$$V^* = Sc^*/G^{1-(n/2)}. \tag{41}$$

Equations (36) and (37) are now linear second order ordinary differential equations with initial conditions. Such equations may be solved readily by using the Runge–Kutta–Nystrom method. This was done using a computer programme which first searched for a suitable time step-length and then used this step-length in proceeding to solve the equations for A vs T with T made to vary over the range 0–100. Various values of the parameters V and ϕ and the power law exponent n were considered in a number of computer solutions.

The results of the numerical solution are presented graphically in Figs. 1–8. They show that: (i) Viscous forces become significant only when V and $V^* > 0.1$ approx., for both Newtonian and power law liquids. As time T increases, the growth rate $\mathrm{d}A/\mathrm{d}T$ approaches a constant value which becomes less for smaller values of V. (ii) The effect of increasing the surface tension parameter ϕ is to stretch the whole growth curve along the time axis. When the surface tension forces are gradually overcome and rendered insignificant, growth is resumed at a rate which would have been achieved had these forces been absent. (iii) All else being equal, the bubble radius and growth rate at a given time T decrease with decreasing power law exponent.

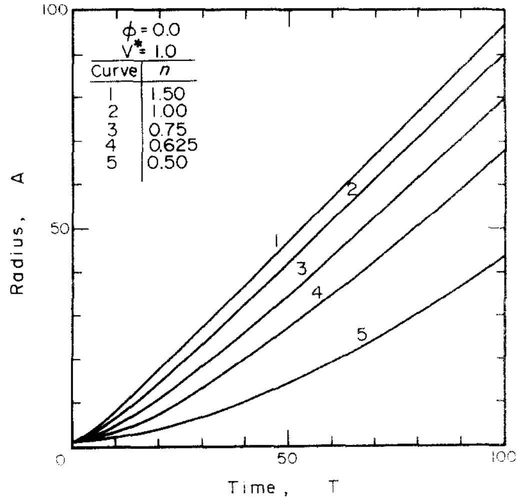


FIG. 1. Growth of a bubble in a power-law fluid of zero surface tension ($\phi = 0$); viscosity/inertia coefficient $V = 1.0$.

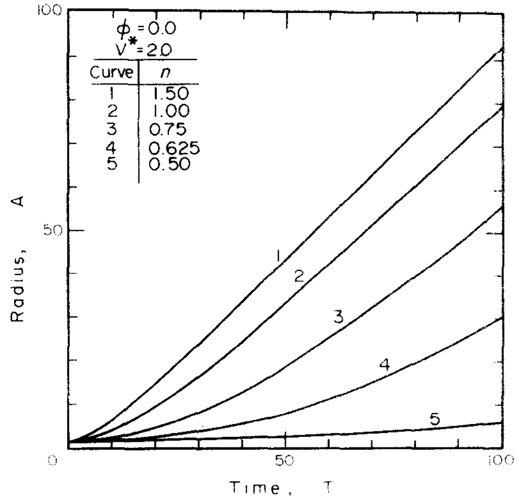


FIG. 2. Growth of a bubble in a power-law fluid of zero surface tension ($\phi = 0$); viscosity/inertia coefficient $V = 2.0$.

(b) *Growth limited by diffusion*

When the properties of the surrounding liquid have no effect on the bubble, its growth rate becomes limited by the rate at which material can diffuse to the bubble surface. The growth is governed solely by the diffusion equation (3) and the mass balance equation (4). This case was solved analytically by Scriven [1] who

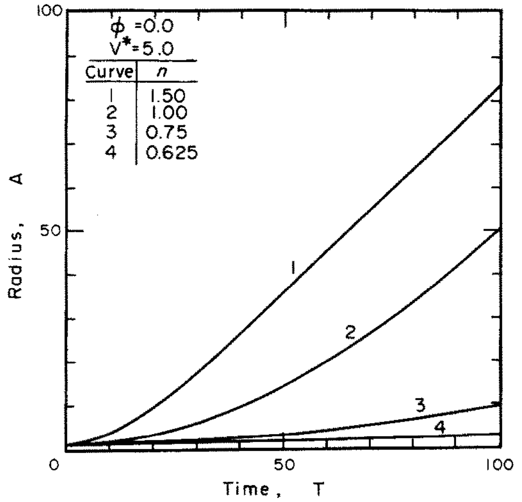


FIG. 3. Growth of a bubble in a power-law fluid of zero surface tension ($\phi = 0$); viscosity/inertia coefficient $V = 5.0$.

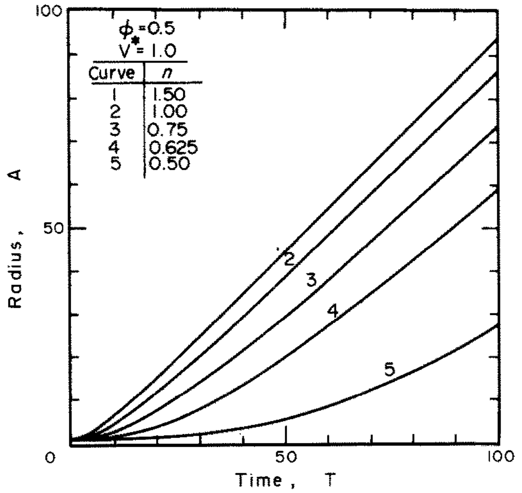


FIG. 4. Hydrodynamically controlled bubble growth in a power-law fluid: $\phi = 0.5$, $V = 1.0$.

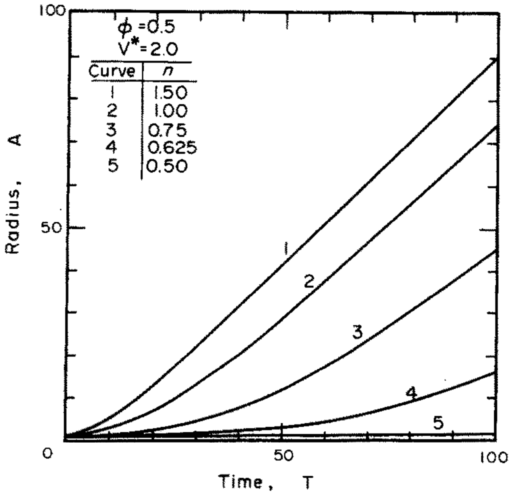


FIG. 5. Hydrodynamically controlled bubble growth in a power-law fluid; $\phi = 0.5$, $V = 2.0$.

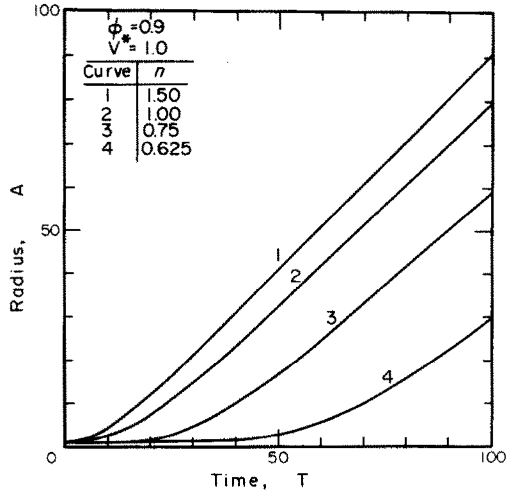


FIG. 6. Hydrodynamically controlled bubble growth in a power-law fluid; $\phi = 0.9$, $V = 1.0$.

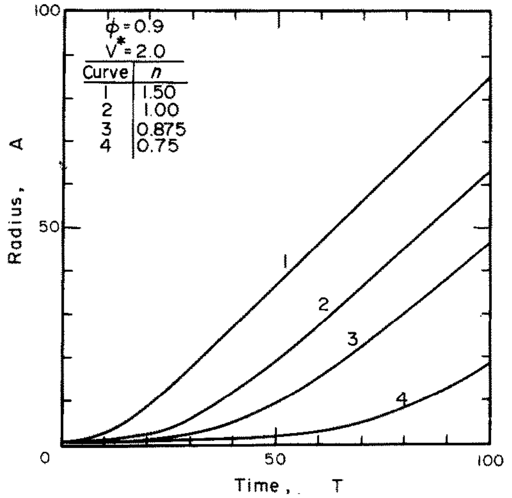


FIG. 7. Hydrodynamically controlled bubble growth in a power-law fluid; $\phi = 0.9$, $V = 2.0$.

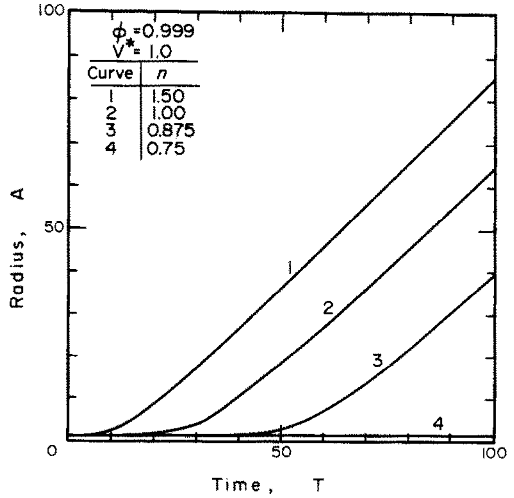


FIG. 8. Growth of a bubble in a power-law fluid with very large surface tension parameter ($\phi = 0.999$); $V = 1.0$.

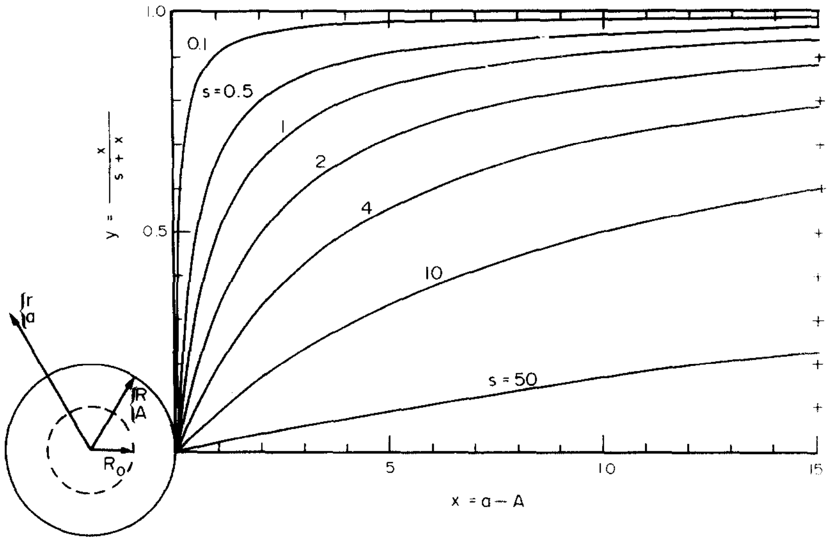


FIG. 9. Nomenclature for co-ordinate transformation, equation (47).

presented an asymptotic solution thus:

$$R = 2\beta(Dt)^{1/2} \tag{42}$$

or in the present dimensionless nomenclature

$$A = 2\beta\tau^{1/2} \tag{43}$$

where β is a "dimensionless growth coefficient" which may be obtained from Tables presented by Scriven or calculated from

$$\begin{aligned}\Phi \equiv \Phi\{\varepsilon, \beta\} &\equiv \frac{\rho_l(C_0 - C_{\text{sat}})}{\rho_g(\rho_l - C_{\text{sat}})} \equiv Ja \\ &= 2\beta^3 \exp(\beta^2 + 2\varepsilon\beta^2) \\ &\quad \times \int_\beta^\infty u^{-2} \exp\left(-u^2 - \frac{2\varepsilon\beta^3}{u}\right) \mathrm{d}u.\end{aligned} \tag{44}$$

For small β ($\beta < 0.025$),

$$Ja \rightarrow 2\beta^2 \tag{45}$$

For large β ($\beta > 8.0$),

$$Ja \rightarrow \beta(\pi/3)^{1/2}. \tag{46}$$

Growth rates calculated from Scriven's solution are plotted as part of Figs. 10–13.

(c) *Growth limited by hydrodynamics and diffusion*

When growth is limited by diffusion as well as by hydrodynamic forces, the concentration of the diffusing species at the bubble surface, and hence the equilibrium gas pressure within the bubble, decreases as the bubble grows. It would be expected that, given long enough times, this mixed case would always approach the solution given for diffusional limitation alone.

So, in this instance, all the equations (26)–(34) will be significant. Now, there are a large number of parameters and variables involved in these equations and it is not possible to reduce their total number. Thus the equations will be left in terms of the first dimensionless time τ, with some changes made to put parameters in terms of their reference or initial values. The radial co-ordinate will also be changed to a more suitable quantity.

The first change of spatial variable is to convert to x, the dimensionless distance beyond the bubble surface. It is proposed to solve the equations numerically. However the x domain is not very convenient for a stepwise integration because the numerical solution has to be evaluated at a fixed number of points whilst the range of x is from zero to infinity. The x domain was thus transformed to a y domain, defined thus:

$$y = x/(s+x). \tag{47}$$

The range is automatically restricted from $y = 0$ (when $x = 0$) to $y = 1$ (when $x = \infty$) and the spacing of the points near the bubble surface is determined by the spacing parameter s. This is well illustrated in Fig. 9.

The equations (26) and (27) remain unchanged but equations (28)–(34) are transformed to:

$$\begin{aligned}\frac{\partial c^*}{\partial \tau} = \frac{(1-y)^4}{s^2}\frac{\partial^2 c^*}{\partial y^2} + \Bigg\{\frac{2(1-y)}{A(1-y)+sy} \\ + \left[1 - \left(\frac{A(1-y)}{A(1-y)+sy}\right)^2\right]\frac{\mathrm{d}A}{\mathrm{d}\tau}\Bigg\}\frac{(1-y)^2}{s}\frac{\partial c^*}{\partial y}\end{aligned} \tag{48}$$

$$A = 1 \qquad \tau = 0 \tag{49}$$

$$\frac{\mathrm{d}A}{\mathrm{d}\tau} = 0 \qquad \tau = 0 \tag{50}$$

$$C^* = 1 \qquad \tau = 0;\ y \geqslant 0 \tag{51}$$

$$C^* = 1 \qquad \text{all } \tau;\ y = 1 \tag{52}$$

$$\left.\frac{1}{s}\frac{\partial c^*}{\partial y}\right|_{y=0} = \frac{1+c^*}{\overline{Ja}}\frac{\mathrm{d}A}{\mathrm{d}\tau} \qquad y = 0 \tag{53}$$

$$\Pi = C^* \qquad y = 0. \tag{54}$$

The Jakob number $\overline{Ja}$ is here defined at the reference pressure P. A further parameter $\bar{B}$ is also defined as

$$\bar{B} = \overline{Ja}^2/G^{1/2}. \tag{55}$$

First suggested by Florschuetz and Chao [7], $\bar{B}$ defines whether a system is in a region of mainly diffusion control ($\bar{B} \ll 1$) or mainly hydrodynamic control ($\bar{B} \gg 1$). In examining the intermediate region, twelve different sets of the parameters $V, G, \overline{Ja}, \bar{B}, \gamma$ and ϕ will be selected. It is an unrealistic task to cover all the parameters in detail and the twelve sets were selected carefully to show the more significant results. Table 1 shows the values of the parameters for the twelve sets. Two Newtonian and two power law cases were considered for each, making 48 solutions.

Table 1. Parameters used in the numerical solutions

Set	$\bar{B}$	$\overline{Ja}$	G	γ	ϕ
1	0.1	100.0	10^{10}	10.0	0.0
2		31.6	10^{8}		
3		10.0	10^{6}		
4	1.0	100.0	10^{8}	10.0	0.0
5		31.6	10^{6}		
6		10.0	10^{4}		
7	10.0	100.0	10^{6}	10.0	0.0
8		31.6	10^{4}		
9		10.0	10^{2}		
10	100.0	100.0	10^{4}	10.0	0.0
11		31.6	10^{2}		
12		10.0	1		

Equation (48), though complicated, is of standard form and is a parabolic, linear partial differential equation. Details of the method of solution are described elsewhere [5]. There are two equations and their associated boundary conditions to be solved together. Equation (26) or (27) is an ordinary differential equation and is solved in each time step by the Runge–Kutta–Nystrom procedure. It is linked to equation (48) at the bubble surface by the boundary condition equation (54).

Initially, Π is equal to unity and the first movement of the bubble surface must be limited by liquid properties. Subsequently the concentration field around the bubble alters. The momentum equation (26) or (27) is first solved to find a new A and $\mathrm{d}A/d\tau$ at the end of a time interval. These new values are then substituted into the diffusion equation (48), and its boundary conditions, which is then solved to find the concentration field at the end of the time interval. This complete solution includes a new value of the interface concentration C_i^*, which equals Π. The new value of Π determines the solution of the momentum equation during the next time step and the procedure is repeated. Considerable numerical experimentation was needed to determine suitable time steps and values of the spacing parameter s.

It was found that the results of the twelve sets of solutions could be adequately represented on four graphs because the shape of the curves (on log–log graph paper) was similar within a pen line thickness. The results are presented as Figs. 10–13.

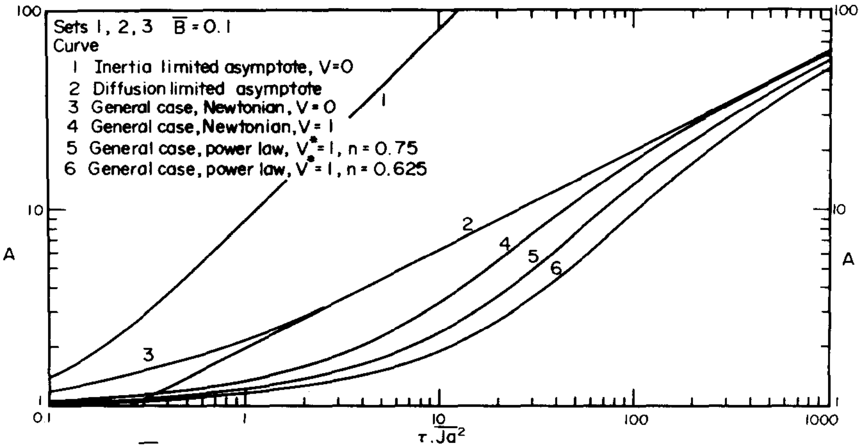
DISCUSSION OF RESULTS

Some of the implications of the results for the general case of growth in Newtonian liquids have already been discussed by Szekely and Martins [2] and Florschuetz and Chao [7]. We summarise their discussion and expand it to the case of a power law liquid here.

(i) The parameter $\bar{B}$ provides a good characterisation of the regime. The larger $\bar{B}$ becomes, the more the system approaches hydrodynamic limitation; the smaller $\bar{B}$ becomes, the more the system is limited by diffusion.

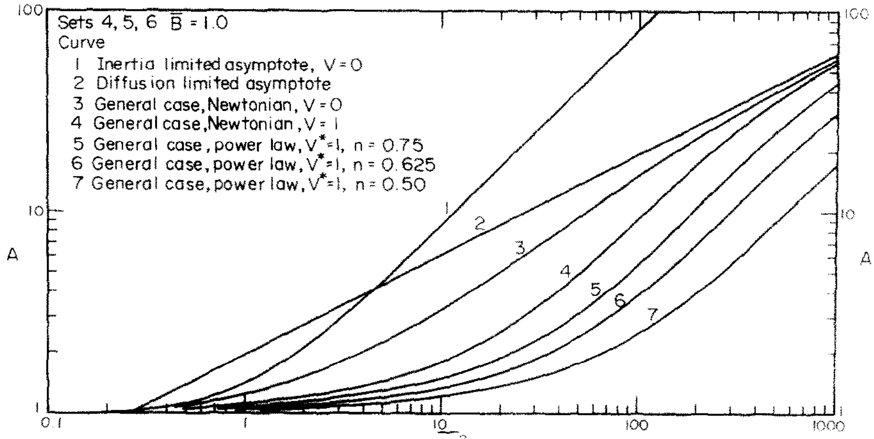
(ii) In the earliest stages of every case, the growth is limited by factors other than diffusion. As time increases, there is a gradual approach to diffusive limitation. The rapidity of the approach depends on the value of $\bar{B}$.

(iii) Provide that $\bar{B} < 0.1$, the time taken to reach a certain size may be estimated without serious error by the asymptotic diffusion limited theory [1].



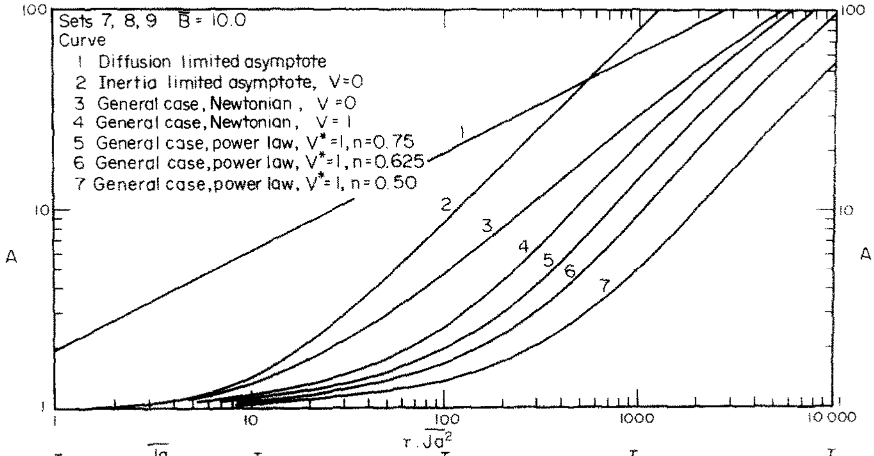
	τ	$\overline{Ja}$	τ	τ	τ	τ
1	10^{-5}	100.0	10^{-4}	10^{-3}	10^{-2}	10^{-1}
2	10^{-4}	31.6	10^{-3}	10^{-2}	10^{-1}	1
3	10^{-3}	10.0	10^{-2}	10^{-1}	1	10

FIG. 10. Bubble growth limited by hydrodynamics and diffusion. Sets 1–3 (Table 1). Diffusion dominant ($\bar{B}$ = 0.1).



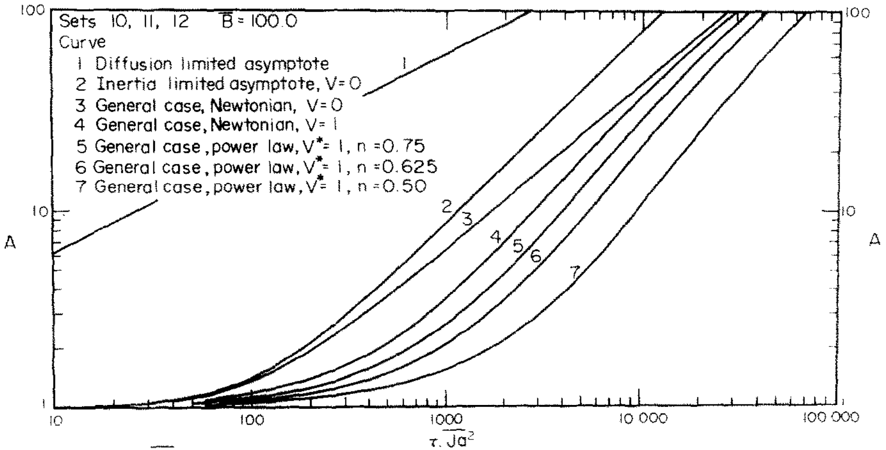
	τ	$\overline{Ja}$	τ	τ	τ	τ
4	10^{-5}	100.0	10^{-4}	10^{-3}	10^{-2}	10^{-1}
5	10^{-4}	31.6	10^{-3}	10^{-2}	10^{-1}	1
6	10^{-3}	10.0	10^{-2}	10^{-1}	1	10

FIG. 11. Bubble growth by hydrodynamics and diffusion. Sets 4–6 (Table 1); $\bar{B} = 1.0$.



	τ	$\overline{Ja}$	τ	τ	τ	τ
7	10^{-4}	100.0	10^{-3}	10^{-2}	10^{-1}	1
8	10^{-3}	31.6	10^{-2}	10^{-1}	1	10
9	10^{-2}	10.0	10^{-1}	1	10	100

FIG. 12. Bubble growth by hydrodynamics and diffusion. Sets 7–9 (Table 1); $\bar{B} = 10.0$.



	τ	$\overline{Ja}$	τ	τ	τ	τ
10	10^{-3}	100.0	10^{-2}	10^{-1}	1	10
11	10^{-2}	31.6	10^{-1}	1	10	100
12	10^{-1}	10.0	1	10	100	1000

FIG. 13. Bubble growth by hydrodynamics and diffusion. Sets 10–12 (Table 1). Hydrodynamics dominant over diffusion ($\bar{B} = 100.0$).

(iv) All else being equal, the growth curve of a power law liquid approaches that of a Newtonian liquid of the same V and ϕ. The rate of growth and the radius at a given time decrease with decreasing power law exponent.

(v) As time proceeds, the solution for a power law liquid approaches that for a Newtonian liquid; they both approach the solution for an inviscid Newtonian liquid then all three approach the diffusion limited asymptote. The convergence to the diffusion limited asymptote is more rapid, the smaller the value of $\bar{B}$.

It is interesting to examine this behaviour of the power law liquid. Its trend is due to the rheological properties of a power law liquid; its magnitude is due partly to the physical properties and partly to the way in which the parameters are defined. When the bubble is just beyond its initial size and growing at a very slow rate, the viscosity of the liquid looks very large to the bubble. Its expansion will thus be substantially retarded. Restating the definitions of the parameters:

$$G = R_0^2 \Delta P/\rho_l D^2 \tag{56}$$

$$Sc^* = mR_0^{2(1-n)}/D^{(2-n)} \tag{57}$$

$$V^* = Sc^*/G^{[1-(n/2)]} \tag{58}$$

$$= \frac{m}{R_0^n}\left(\frac{\rho_l}{\Delta P}\right)^{(2-n)/2}. \tag{59}$$

The density of most non-Newtonian liquids is close to that of water so ρ_l can be regarded effectively as constant. Thus e.g. (59) can be written:

$$V^* \propto \frac{1}{R_0^n}\left(\frac{1}{\Delta P}\right)^{(2-n)/2}. \tag{60}$$

Now, if the initial radius is that which can exist in unstable equilibrium with the supersaturated solution then, through the surface tension relationship:

$$R_0 \propto 1/\Delta P. \tag{61}$$

Substituting in (60) gives:

$$V^* \propto (R_0)^{1-(3n/2)}. \tag{62}$$

The expression illustrates the strong dependence of V^* on the exponent n, for a given R_0, assuming that R_0 and ΔP are inversely proportional. This is a weakness in the suitability of the parameter but nevertheless the conclusion still holds that, for equal G, V and ϕ, bubble radius and growth rate at a given time decrease with decreasing power law exponent. It is not possible to say more than that in a general way, and it is a valid result.

Note that the dimensionless time T and the dimensionless radius A also include the value of R_0, viz.:

$$T \propto R_0^{-3/2} t \tag{63}$$

$$A = R_0^{-1} R. \tag{64}$$

So all the parameters V^*, G, ϕ, T and A depend on the value of R_0 and V^* on n as well. That this is not as serious a weakness as it might seem is shown by Szekely and Martins [2]. In calculations based on experimental results in a Newtonian liquid, they used $R_0 = 10^{-2}$, 10^{-3} and 10^{-4} cm and showed that the actual value chosen for R_0 did not materially affect the actual radius vs real time curve.

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CROISSANCE DES BULLES SPHERIQUES PAR DIFFUSION DE SOLUTE DANS LES LIQUIDES NON NEWTONIENS (LOI PUISSANCE)

Résumé—On étudie la croissance des bulles dans des liquides newtoniens et non newtoniens du type loi puissance. On considère trois régions, la première dans laquelle la croissance est limitée entièrement par la vitesse d'arrivée du gaz dissous à la surface de la bulle, la seconde où la croissance est entièrement limitée par des forces hydrodynamiques et la troisième, un cas général, pour laquelle la diffusion et l'hydrodynamique jouent ensemble. Des paramètres qui ont été précédemment utilisés pour définir des régionsoù la viscosité, l'inertie, la tension de surface et la diffusion sont importantes, restent encore valables pour des liquides à loi puissance. On trouve que, toute chose égale par ailleurs, les bulles dans des liquides à loi puissance croissent plus lentement lorsque l'exposant caractéristique est plus faible.

WACHSTUM VON KUGELFÖRMIGEN GASBLASEN DURCH DIFFUSION VON GELÖSTEM GAS IN NICHT-NEWTONSCHEN (POTENZ-GESETZ-) FLüSSIGKEITEN

Zusammenfassung—In dieser Arbeit wird das Blasenwachstum durch Diffusion in Newtonschen und Nicht-Newtonschen Flüssigkeiten des Potenz-Gesetz-Typs untersucht. Drei Bereiche werden betrachtet, im ersten ist das Blasenwachstum vollständig durch die Geschwindigkeit des an die Blasenoberfläche gelangenden gelösten Gases bestimmt, im zweiten wird das Wachstum durch hydrodynamische Kräfte beeinflußt, im dritten, einem allgemeinen Fall spielen sowohl die Diffusion als such hydrodynamische Effekte eine Rolle. Die früher verwendeten Parameter zur Kennzeichnung der verschiedenen Bereiche, wie Viskosität, Trägheit, Oberflächenspannung und Diffusion haben auch bei Potenz-Gesetz-Flüssigkeiten Gültigkeit. Es hat sich herausgestellt, daß unter sonst gleichen Bedingungen Blasen in Potenz-Gesetz-Flüssigkeiten mit abnehmendem Exponenten langsamer wachsen.

РОСТ СФЕРИЧЕСКИХ ПУЗЫРЬКОВ ГАЗА ПРИ ДИФФУЗИИ РАСТВОРЕННОГО ВЕЩЕСТВА В НЕНЬЮТОНОВСКИХ ЖИДКОСТЯХ, ОПИСЫВАЕМЫХ СТЕПЕННЫМ ЗАКОНОМ

Аннотация — Исследуется диффузионный рост пузырьков газа в ньютоновских и неньютоновских жидкостях, описываемых степенным реологическим законом. Рассматриваются три области: в первой области рост пузырьков ограничен только скоростью поступления растворенного газа на поверхность раздела; во второй, рост пузырьков ограничен только гидродинамическими силами; и в третьей, представляющей собой общий случай, влияние как диффузии, так и гидродинамики существенно. Показано, что параметры, использованные ранее для определения областей, в которых важное значение имеют вязкость, инерция, поверхностное натяжение и диффузия, могут быть применены для жидкостей, подчиняющихся степенному закону. Установлено, что при прочих равных условиях, пузырьки в неньютоновских жидкостях растут тем медленнее, чем меньше индекс течения.